The World of q

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A few of the many ways in which q-series arise in mathematics will be considered. These include a discrete approximation to the normal integral, Ramanujan's extension of the beta integral on $(0, \infty)$, the q-integral which was introduced by Fermat to integrate the power function but has many other uses, and the continuous q-ultraspherical polynomials of L. J. Rogers. Here an absolutely continuous measure occurs in the orthogonality.

1. Introduction

The first explicit connection between group theory and special functions which most mathematicians see is trigonometric functions as a parametrization of the unit circle, with the addition formula for $\cos(\theta + \varphi)$ coming from the rotation of the circle and the fact that distances are preserved under this rotation. For most mathematicians this is the only explicit connection they have seen between group theory and special functions, but they have some acquaintance with a few other functions which can be represented as hypergeometric functions, so when they are told that there are functions similar to $\cos\theta$ and $\sin\theta$ which arise in the representation theory of some other groups they are willing to believe that this might be important, and that if they spent a reasonable amount of time they could learn this material.

Basic hypergeometric functions are a different matter. Here, most mathematicians say they have never seen any of these functions, and most of them are right. Klimyk (this issue) has surveyed much of Tom Koornwinder's work. Recently this work has dealt with basic hypergeometric functions, usually in connection with quantum groups. Noumi (this issue) has written about quantum groups and some of the ways basic hypergeometric functions arise there. I want to start at a more elementary level and show how some basic hypergeometric functions arose earlier. In keeping with the historical development, and so as not to frighten off readers with notation which seems cumbersome, basic

To Tom Koornwinder with thanks for what he has taught us hypergeometric functions will not be considered in detail and the standard notation for the terms in these series will only be introduced later. For the present, hypergeometric and basic hypergeometric series can be thought of as follows.

The favorite test for convergence of an infinite series for students just learning about convergence is the ratio test. If the series

$$\Sigma c_n \tag{1.1}$$

has

$$\frac{c_{n+1}}{c_n} = \text{rational function of } n, \tag{1.2}$$

the series is a hypergeometric series. If

$$\frac{c_{n+1}}{c_n} = \text{rational function of } q^n, \tag{1.3}$$

where q is a fixed number, $|q| < 1$, the series is called a basic hypergeometric series.

2. A discrete approximation to the normal integral

The normal integral is

$$\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}\,. \tag{2.1}$$

One approximation to this is obtained by taking a set of equally spaced points. There is no reason why this set needs to contain 0, so this approximation is

$$\tau^{\frac{1}{2}} \sum_{-\infty}^{\infty} e^{-\tau(n+c)^2} = \tau^{\frac{1}{2}} e^{-\tau c^2} \sum_{-\infty}^{\infty} e^{-\tau n^2} e^{-2\tau c n} \tag{2.2}$$

The series on the right is what is usually called a theta function. There are a number of questions one can ask about this function or the series on the left.

How rapidly does the series on the left converge to the normal integral (2.1) when $\tau \to 0$? (2.3)

What are the zeros of this function? (2.4)

What can we use this function for? (2.5)

We start with question (2.3), and rewrite the left hand side as

$$f(x) = \tau^{\frac{1}{2}} \sum_{-\infty}^{\infty} e^{-\tau(n+x)^2}, \qquad \tau > 0\,. \tag{2.6}$$

The big difference between an integral and a series is that it is possible to change variables in an integral, but not in a series. The one change of variables which can be done with the series (2.6) is to shift the parameter n by an integer. This leads to the important, but a first sight trivial result

$$f(x+1) = f(x)\,. \tag{2.7}$$

Once this is observed, it is natural to expand $f(x)$ in a Fourier series

$$f(x) = \sum_{-\infty}^{\infty} a_k e^{2\pi i k x}\,. \tag{2.8}$$

Then

$$\begin{aligned} a_k &= \int_0^1 \tau^{\frac{1}{2}} \sum_{-\infty}^{\infty} e^{-\tau(n+x)^2} e^{-2\pi i k x} dx \\ &= \tau^{\frac{1}{2}} \sum_{\pi=-\infty}^{\infty} e^{\frac{-k^2\pi^2}{\tau}} \int_n^{n+1} e^{-\tau(x+\frac{\pi i k}{\tau})^2} dx \\ &= \tau^{\frac{1}{2}} e^{-k^2\pi^2/\tau} \int_{-\infty}^{\infty} e^{-\tau(x+\frac{\pi i k}{\tau})^2} dx \\ &= \tau^{\frac{1}{2}} e^{-k^2\pi^2/\tau} \left(\frac{\pi}{\tau}\right)^{\frac{1}{2}} \end{aligned}$$

by shifting the line of integration and using Cauchy's theorem and an easy estimate on the decay of the integrand. Thus

$$\sum_{-\infty}^{\infty} e^{-\tau(n+x)^2} = \left(\frac{\pi}{\tau}\right)^{\frac{1}{2}} \sum_{-\infty}^{\infty} e^{-k^2\pi^2/\tau + 2\pi i k x}\,. \tag{2.9}$$

We have proved (2.9) when $\tau > 0$, but the proof continues to hold when Re $\tau > 0$ and the principal branch of the square root is taken. To answer the question of how rapidly (2.2) converges to (2.1), rewrite (2.9) as

$$\sqrt{\tau} \sum_{-\infty}^{\infty} e^{-\tau(n+c)^2} = \sqrt{\pi}\left[1 + 2\sum_{k=1}^{\infty} e^{-k^2\pi^2/\tau} \cos 2\pi c k\right]\,. \tag{2.10}$$

This is an incredible formula. It gives the error rate of $O(e^{-\pi^2/\tau})$, which for $\tau = .01$ gives a bound of size $e^{-100\pi^2}$ which is less than 10^{-428}. However it gives much more, a complete series giving as many terms as one wants as in a convergent series.

What we have done above is to give the usual argument to prove the Poisson summation formula, but for one specific function. The same argument works as long as the function $f(x)$ decreases rapidly enough and its Fourier series converges absolutely, which is essentially determined by the smoothness of $f(x)$.

Formula (2.10) was discovered by Poisson and quite a few others. BELLMAN [7] gives or outlines four or five different proofs and there are others.

Question (2.4) may seem like a natural question when you are starting to study mathematics, but after initial disappointments one usually learns to ask a much less ambitious question, such as, what can one say about the zeros of the function in question? Recall that the Riemann hypothesis for the zeta function only says that the nontrivial zeros lie on a line. They can not be determined exactly. So what gives us the idea that we can hope to find the zeros of the function exactly? The periodicity above says that if one zero can be found then an infinite number of them are known. That is not sufficient evidence to lead us to suspect that any of the zeros can be found exactly, but the incredible formula given in (2.10) might lead us to expect more to be possible with this function than with most periodic functions. My old teacher, S. Bochner, used to refer to this topic as the miracle of theta functions, and that is not an overstatement.

To consider the zeros we start with new notation. Consider

$$f(x,q) = \sum_{-\infty}^{\infty} q^{n^2} x^n \tag{2.11}$$

where $|q| < 1$ will be assumed from now unless we state differently. The shift in n used above gives

$$f(x,q) = \sum_{-\infty}^{\infty} q^{(n+1)^2} x^{n+1} = qxf(q^2x,q)\,. \tag{2.12}$$

If $f(x,q)$ vanishes for one value x_0, it must vanish on the bilateral geometric sequence

$$x_0 q^{2n}\,, \qquad n = 0, \pm 1, \pm 2, \ldots\,. \tag{2.13}$$

Rather than try to find x_0 directly, and then show that all the zeros are given by the bilateral geometric sequence (2.13), we start with one zero and write a function which has zeros on exactly this bilateral geometric sequence.

The most natural function which vanishes on the integers is $\sin \pi x$. Euler showed that

$$\sin \pi x = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)\,. \tag{2.14}$$

To find an analogous function whose zeros lie on a geometric sequence consider a product which corresponds to (2.14). Just as in (2.14), we had to combine factors to get a convergence product, since

$$\prod_{n=1}^{\infty} \left(1 - \frac{x}{n}\right) \tag{2.15}$$

diverges, we need to do a bit more than write down the product of terms of the form $(1-(xq^n/x_0))$, $n=0,\pm 1,\dots$. This works for $n \geq 0$, but if all the terms for $n<0$ are used the product diverges. Consider

$$h(x,q)=\prod_{n=0}^{\infty}(1+q^{2n+1}x)(1+q^{2n+1}x^{-1})\,. \tag{2.16}$$

This vanishes when $x=-q^{1+2n}$, $n=0,\pm 1,\pm 2,\dots$, and is analytic in x for all complex x except $x=0$. Thus $h(x,q)$ can be expanded in a Laurent series

$$h(x,q)=\sum_{-\infty}^{\infty}c_n x^n\,. \tag{2.17}$$

As is suggested by (2.12), observe that

$$\frac{h(q^2x,q)}{h(x,q)}=\frac{\left(1+\frac{1}{qx}\right)}{1+qx}=\frac{1}{qx} \tag{2.18}$$

or

$$\sum_{-\infty}^{\infty}c_n x^n = qx\sum_{-\infty}^{\infty}c_n q^{2n}x^n\,. \tag{2.19}$$

Equating coefficients of x^{n+1} gives

$$c_{n+1}=q^{2n+1}c_n\,. \tag{2.20}$$

Thus

$$c_n=q^{n^2}c_0\,,\qquad n=0,1,\dots\,. \tag{2.21}$$

From

$$h(x;q)=h(x^{-1},q) \tag{2.22}$$

we get

$$c_{-n}=c_n \tag{2.23}$$

so (2.21) continues to hold when $n=-1,-2,\dots$. Thus

$$\prod_{n=0}^{\infty}(1+q^{2n+1}x)(1+q^{2n+1}x^{-1})=c_0\sum_{-\infty}^{\infty}q^{n^2}x^n\,. \tag{2.24}$$

To find c_0, a different argument must be given. Jacobi sieved the series using specific values of x, say $x=q$ and $x=-q$. This gives

$$\sum_{-\infty}^{\infty} q^{4n^2+2n} = [c_0(q)]^{-1}\prod_{n=1}^{\infty}(1+q^{2n})^2 = [c_0(q^4)]^{-1}\prod_{n=0}^{\infty}(1+q^{8n+2})(1+q^{8n+6}) \tag{2.25}$$

so

$$\begin{aligned}\frac{c_0(q)}{c_0(q^4)} &= \prod_{n=0}^{\infty}\frac{(1+q^{2n+2})^2(1+q^{8n+4})(1+q^{8n+8})}{(1+q^{8n+2})(1+q^{8n+6})(1+q^{8n+4})(1+q^{8n+8})} \\ &= \prod_{n=0}^{\infty}(1+q^{2n+2})(1+q^{4n+4}) \\ &= \prod_{n=0}^{\infty}\frac{(1-q^{4n+4})(1+q^{4n+4})}{(1-q^{2n+2})} \\ &= \prod_{n=1}^{\infty}\frac{(1-q^{8n})}{(1-q^{2n})}\,.\end{aligned}$$

Thus

$$c_0(q)\prod_{n=1}^{\infty}(1-q^{2n}) = c_0(q^4)\prod_{n=1}^{\infty}(1-q^{8n}) = c_0(q^{2^k})\prod_{n=1}^{\infty}(1-q^{n2^{k+1}}) = c_0(0) = 1 \tag{2.26}$$

so

$$\sum_{-\infty}^{\infty} q^{n^2}x^n = \prod_{n=0}^{\infty}(1-q^{2n+2})(1+q^{2n+1}x)(1+q^{2n+1}x^{-1})\,. \tag{2.27}$$

This is usually known as the triple product for obvious reasons.

Replacing q by $q^{\frac{3}{2}}$ and then taking $x = -q^{-\frac{1}{2}}$ gives Euler's formula

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{-\infty}^{\infty}(-1)^n q^{\frac{3n^2}{2}-\frac{n}{2}}\,. \tag{2.28}$$

There are other ways to find $c_0(q)$. One of these, which leads to a more general identity of Ramanujan, will be considered in the next section.

3. The q-binomial theorem and its extension by Ramanujan

The problem in trying to find $c_0(q)$ in (2.24) could be solved easily if there were a value of x for which one could evaluate the series in (2.24). Over one hundred years ago, Franklin [12] gave a simple combinatorial proof of (2.28). However there are no values of x for which it is obvious how to sum the series in (2.24). Since being greedy has paid off so far, we can try to be greedy again and consider a function whose zeros are on part of bilateral geometric series. We can take one of the products in (2.16), as Euler did. We can also consider finite products, taking the first factors from each of the two products. These can be taken symmetrically, which is what is done in several variables, but here there is no need to since any finite string of products can be moved by sliding them from one product to the other one. Also, there is no longer any advantage in using q^2 rather than q as the factor between zeros, so we will use q. With this change we can consider the partial product

$$f_{m,n}(x) = \prod_{k=0}^{n-1}(1 - xq^k) \prod_{k=0}^{m-1}(1 - q^{k+1}x^{-1}) \tag{3.1}$$

where we have taken a different choice for the variable to simplify the final formula. It is now time to introduce the standard notation:

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1 - xq^k) \tag{3.2}$$

$$(x;q)_n = \frac{(x;q)_\infty}{(xq^n;q)_\infty} = \prod_{k=0}^{n-1}(1 - xq^k) \tag{3.3}$$

The product in (3.1) can be written as

$$f_{m,n}(x) = (x;q)_n(qx^{-1};q)_m = \frac{(x;q)_\infty(qx^{-1};q)_\infty}{(xq^n;q)_\infty(q^{m+1}x^{-1};q)_\infty}. \tag{3.4}$$

This product is a Laurent polynomial in x, and so it can be written as

$$f_{m,n}(x) = \sum_{k=-m}^{n} c_k x^k . \tag{3.5}$$

The same type of argument used in the previous section gives

$$\frac{f_{m,n}(qx)}{f_{m,n}(x)} = \frac{(1 - xq^n)(1 - x^{-1})}{(1 - x)(1 - q^m x^{-1})} = \frac{(1 - q^n x)}{(q^m - x)}. \tag{3.6}$$

Observe the cancellation of the factors $1 - x^{-1}$ and $1 - x$. This occurs because the numerator in the right-hand equality in (3.4) is a theta product. By this we mean that there are two products

$$(a;q)_\infty(b;q)_\infty \tag{3.7}$$

and the product of the variables $a \cdot b$ is the base q. Recall (2.27), which we now write as

$$(q^2;q^2)_\infty(-qx;q^2)_\infty(-qx^{-1};q^2)_\infty = \sum_{-\infty}^{\infty} q^{n^2}x^n\,. \tag{3.8}$$

The factor which depend on x have the property that the product of these variables, $(-qx)(-qx^{-1}) = q^2$, is the base. That is why we call the products in (3.7) a theta product when $ab = q$. The third factor in (3.8) is important, but we want a way to distinguish the case of three products, which we call the triple product, from the case of two products which lead to a theta function as a Laurent series in x.

The functional equation (3.6) gives a first order recurrence relation for the coefficients c_k. It is

$$c_{k+1} = \frac{(q^n - q^k)}{(1-q^{m+k+1})}c_k = \frac{(1-q^{-n+k})q^n}{(1-q^{m+1+k})}c_k\,. \tag{3.9}$$

This is solved by

$$c_k = \frac{(q^{-n};q)_k}{(q^{m+1};q)_k}q^{nk}c_0 \tag{3.10}$$

as is obvious from $k = 1, 2, \ldots, n$ and can be checked easily for $k = -1, -2, \ldots, -m$ by using the definition (3.3) for these q-shifted factorials when $k < 0$. Thus

$$(x;q)_n(qx^{-1};q)_m = c_0 \sum_{k=-m}^{n} \frac{(q^{-n};q)_k}{(q^{m+1};q)_k}(q^n x)^k\,. \tag{3.11}$$

To find c_0 is now easy, since the coefficient of x^n on both sides is easy to find. These coefficients give

$$(-1)^n q^{n(n-1)/2} = \frac{c_0(q^{-n};q)_n}{(q^{m+1};q)_n}q^{n^2} \tag{3.12}$$

or

$$c_0 = \frac{(q^{m+1};q)_n}{(q;q)_n}\,. \tag{3.13}$$

Thus

$$\frac{(x;q)_n(qx^{-1};q)_m(q;q)_n}{(q^{m+1};q)_n} = \sum_{k=-m}^{n} \frac{(q^{-n};q)_k}{(q^{m+1};q)_k}(q^n x)^k\,. \tag{3.14}$$

When $m = 0$ this is an extension of the binomial theorem to

$$(x;q)_n = \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k}(q^n x)^k\,, \tag{3.15}$$

or replacing x by $-x$ we have

$$\prod_{k=0}^{n-1}(1+xq^k)=\sum_{k=0}^{n}\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}q^{k(k-1)/2}x^k\,. \tag{3.16}$$

The factor

$$\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}=\begin{bmatrix}n\\k\end{bmatrix}_q \tag{3.17}$$

is called the q-binomial coefficient, or the Gaussian polynomial. It is a polynomial in q with positive integer coefficients whose sum is the binomial coefficient

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}\,. \tag{3.18}$$

When $m\to\infty$ and $n\to\infty$ in (3.11) the result is the triple product in the form

$$(q;q)_\infty(x;q)_\infty(qx^{-1};q)_\infty=\sum_{-\infty}^{\infty}(-1)^n q^{n(n-1)/2}x^n\,. \tag{3.19}$$

Ramanujan's extension of this and the classical q-binomial theorem (3.15) is the nonterminating form of (3.14). This can be found directly from (3.14) after it has been rewritten. The first problem is to remove the q^n in $q^n x$. This is done by replacing x by $q^{-n}x$. Then (3.14) is

$$\sum_{k=-m}^{n}\frac{(q^{-n};q)_k}{(q^{m+1};q)_k}x^k=\frac{(xq^{-n};q)_\infty(\frac{q^{n+1}}{x};q)_\infty(q;q)_\infty(q^{m+n+1};q)_\infty}{(x;q)_\infty(\frac{q^{m+n+1}}{x};q)_\infty(q^{n+1};q)_\infty(q^{m+1};q)_\infty}\,. \tag{3.20}$$

When $q^{m+1}=b$, $q^{-n}=a$, this can be written as

$$\sum_{-\infty}^{\infty}\frac{(a;q)_k}{(b;q)_k}x^k=\frac{(ax;q)_\infty(\frac{q}{ax};q)_\infty(q;q)_\infty(\frac{b}{a};q)_\infty}{(x;q)_\infty(\frac{b}{ax};q)_\infty(b;q)_\infty(\frac{q}{a};q)_\infty}\,. \tag{3.21}$$

where the series is taken over all integers, which can be done since the terms in (3.21) vanish when $k>n$ and when $k<-m$. Formula (3.21) is in fact true when it converges, which it does when

$$\left|\frac{b}{a}\right|<|x|<1\,, \tag{3.22}$$

and it is the identity of Ramanujan referred to above. There are many proofs of (3.21). One particularly nice one is to start with the product side, assume in addition to (3.22) that

$$\left|\frac{b}{aq}\right| < |x|. \tag{2.23}$$

and carry out the argument above. This gives the products on the right hand side which depend on x, and the remaining products can be obtained from Abel's continuity theorem, i.e. multiply both sides by $(1-x)$ and take the limit as $x \to 1^-$, using

$$\lim_{x\to 1^-} f(x) = f(1) \tag{3.24}$$

when $f(z)$ is analytic for $c \le |z| < 1$. The details are given in [1] and [8], which are easily accessible, and an earlier treatment was given in [25]. The idea of proving a special case of (3.21) and using it to get the general result was first given in [14]. Ismail used the classical case when $b = q$ to derive (3.21). Both sides of (3.21) are analytic in b for b sufficiently close to the origin.

An interesting exercise is to take the reciprocal of the products in (3.21) which involve x and apply the above argument to this function. The interest is in trying to find out why a completely erroneous result arises, and then seeing why the above argument is correct.

Question (2.5) has not really been answered sufficiently, so let us give another application. Here we will start to consider q as the primary variable and x as a parameter. Consider the case $b = aq$ of (3.21). The result is

$$\sum_{-\infty}^{\infty} \frac{x^n}{1-aq^n} = \frac{(ax;q)_\infty \left(\frac{q}{ax};q\right)_\infty (q;q)_\infty^2}{(x;q)_\infty \left(\frac{q}{x};q\right)_\infty (a;q)_\infty \left(\frac{q}{a};q\right)_\infty}. \tag{3.25}$$

There are three theta products on the right hand side, which will be interesting in the next section. Here we will do something else with this identity as it relates to theta functions. Take $x = 1$ in (2.27) to get

$$\begin{aligned}\sum_{-\infty}^{\infty} q^{n^2} &= (q^2;q^2)_\infty(-q;q^2)_\infty^2 \\ &= \frac{(q^2;q^2)_\infty(-q;q^2)_\infty(q^2;q^4)_\infty(q^4;q^4)_\infty}{(q;q^2)_\infty(q^4;q^4)_\infty} \\ &= \frac{(q^2;q^2)_\infty(-q;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}.\end{aligned} \tag{3.26}$$

The square of this is

$$\left(\sum_{-\infty}^{\infty} q^{n^2}\right)^2 = \sum_{n=0}^{\infty} r_2(n)q^n \tag{3.27}$$

where $r_2(n)$ is the number of representations of n as the sum of two squares, where each point on the circle

$$j^2 + k^2 = n \tag{3.28}$$

is counted. Thus

$$\begin{aligned}\sum_{n=0}^{\infty} r_2(n)q^n &= \frac{(q^2;q^2)_\infty^2(-q;q^2)_\infty^2}{(q;q^2)_\infty^2(-q^2;q^2)_\infty^2} \\ &= \frac{2(q^2;q^2)_\infty^2(-q;q^2)_\infty^2}{(q;q^2)_\infty^2(-1;q^2)_\infty(-q^2;q^2)_\infty} \\ &= 2\sum_{-\infty}^{\infty}\frac{q^n}{1+q^{2n}} = 1 + 4\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}} \\ &= 1 + 4\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(-1)^k q^{(2k+1)n}.\end{aligned} \tag{3.29}$$

This gives Jacobi's result that

$$r_2(n) = 4[d_1(n) - d_3(n)], \quad n \geq 1 \tag{3.30}$$

where $d_i(n)$ is the number of divisors of n of the form $4m + i$. FINE [11] was the first to use Ramanujan's sum (3.21) to obtain results on sums of squares, although the use of the Lambert series in (3.29) to (3.30) goes back to Jacobi. He had another method of deriving this result.

Finally, there is a noncommutative version of the finite binomial theorem which plays a central role in the development of $\mathrm{SU}_q(2)$, the quantum group analogue of SU(2).

Let x and y be indeterminants a and q a number which satisfy the following rules for multiplication

$$yx = qxy, \quad qx = xq, \quad qy = yq. \tag{3.31}$$

Then

$$(x+y)^n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k}y^k. \tag{3.32}$$

SCHUTZENBERGER [21] stated and proved this result in the form given here, but it is just a very elegant and simple way to state and derive one of MacMahon's statistics for the Gaussian polynomial defined in (3.17). The statistic is the number of inversions. The product rule for exponentials continues to hold in this setting, and when it is used, and the resulting product is expanded directly and as the product of two series, and coefficients are compared, the result is a known q-extension of the Chu-Vandermonde sum for binomial coefficients

$$\sum_{k=0}^{n}\binom{k+a}{k}\binom{n-k+b}{n-k} = \binom{n+a+b}{n}. \tag{3.33}$$

This identity is enough to get started deriving almost all of the known hypergeometric series summations and transformations, and the same is true for the corresponding q-series identity and the known basic hypergeometric summations and transformations.

4. The q-integral

Fermat read Archimedes carefully and extracted a new method of evaluating

$$\int_0^1 x^k dx\,. \tag{4.1}$$

He decomposed the interval $[0,1]$ on a geometric progression, q^n, $n=0,1,\ldots$, using the measure of the interval $[q^{n+1},q^n]$ as a point mass at q^n, and arrived at the following geometric series which he could sum

$$(1-q)\sum_{n=0}^{\infty} q^{nk}q^n = \frac{(1-q)}{1-q^{k+1}} = \frac{1}{1+q+\cdots+q^k}\,. \tag{4.2}$$

Thus he arrived at a proof of

$$\int_0^1 x^k dx = \frac{1}{k+1} \tag{4.3}$$

which had earlier been evaluated for $k=0,1,\ldots,9$. See Toeplitz [24] for a treatment of Archimedes' work and also that of Fermat. In the last century Thomae [23] and later Jackson [15] considered an extension of Fermat's sum, but held q fixed. I like to call the measure used the Fermat measure, which puts mass $(1-q)q^n$ at $x=q^n$, or more generally mass $a(1-q)q^n$ at $x=aq^n$. Then define

$$\int_0^a f(x)d_qx := a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n \tag{4.4}$$

which defines d_qx on this interval. Unless otherwise specified we will take $a=1$.

The nonterminating form of the q-binomial theorem, i.e. $b=q$ of (3.21), can be rewritten as a q-integral. It is

$$\int_0^1 x^{\alpha-1}\frac{(xq;q)_\infty}{(xq^\beta;q)_\infty}d_qx = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} \tag{4.5}$$

where

$$\Gamma_q(\alpha) = \frac{(q;q)_\infty}{(q^\alpha;q)_\infty}(1-q)^{1-\alpha} \tag{4.6}$$

is the natural q-extension of the gamma function. This is easily seen to be a q-extension of the beta integral on $[0,1]$.

To see what Ramanujan's sum (3.21) is, rewrite it as

$$(1-q)\sum_{-\infty}^{\infty}\frac{(-cq^{n+\alpha+\beta};q)_\infty}{(-cq^n;q)_\infty}q^{\alpha n} \tag{4.7}$$

$$= \frac{\Gamma_q(\alpha)\Gamma_q(\beta)(-q^\alpha c;q)_\infty(-q^{1-\alpha}/c;q)_\infty}{\Gamma_q(\alpha+\beta)(-c;q)_\infty(-q/c;q)_\infty}.$$

To see what this extends, take $a = q^{-N}$ in (4.4) and write it as

$$\int_0^{q^{-N}} f(x)d_qx = (1-q)\sum_{n=-N}^{\infty} f(q^n)q^n. \tag{4.8}$$

Then $N \to \infty$ gives a definition for a q-integral on $(0,\infty)$,

$$\int_0^{\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n. \tag{4.9}$$

Thus (4.6) is

$$\int_0^\infty \frac{t^{\alpha-1}(-ctq^{\alpha+\beta};q)_\infty}{(-ct;q)_\infty}d_qt = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)(-cq^\alpha;q)_\infty(-q^{1-\alpha}/c;q)_\infty}{\Gamma_q(\alpha+\beta)(-c;q)_\infty(-q/c;q)_\infty}. \tag{4.10}$$

Then $q \to 1^-$ gives

$$\int_0^\infty \frac{t^{\alpha-1}dt}{(1+ct)^{\alpha+\beta}} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}(1+c)^{-\alpha}(1+c^{-1})^\alpha = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)c^\alpha} \tag{4.11}$$

since

$$\lim_{q\to 1^-}\Gamma_q(\alpha) = \Gamma(\alpha) \tag{4.12}$$

and

$$\lim_{q\to 1^-}\frac{(q^a x;q)_\infty}{(x;q)_\infty} = (1-x)^{-\alpha}, \quad x \text{ not on } [1,\infty). \tag{4.13}$$

For proofs of these two limit relations see Koornwinder [16].

The special case of (4.10) when $\alpha+\beta = 1$ is very important, and is usually written as

$$\int_0^\infty \frac{t^{\alpha-1}}{1+t}dt = \frac{\pi}{\sin \pi\alpha}, \quad 0 < \text{Re } \alpha < 1. \tag{4.14}$$

The analogous q-integral is (3.25). Any book on elliptic functions which includes much about Jacobi elliptic functions has some of the Fourier series expansions of the Jacobi elliptic functions sn, cn, dn, etc. All of these are special cases of (3.25), and so are seen to be q-extensions of Euler's integral (4.6).

5. The continuous q-ultraspherical orthogonal polynomials

These polynomials were introduced by L. J. Rogers in a special case in 1894 [19] and in the general case in 1895 [20]. They were then ignored for decades until they were rediscovered in 1941 by FELDHEIM [10] and LANZEWIZKY [18]. Their motivation for finding these polynomials is easy to explain, so here it is.

Legendre polynomials $P_n(x)$ can be defined by the generating function

$$(1-2xr+r^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) r^n, \quad |r|<1, \quad -1 \le x \le 1. \tag{5.1}$$

When $x = \cos\theta$ this can be rewritten as

$$|(1-re^{i\theta})^{-\frac{1}{2}}|^2 = \sum_{n=0}^{\infty} P_n(\cos\theta) r^n, \quad -1<r<1. \tag{5.2}$$

Fejér considered an extension of these polynomials by looking at a function $f(z)$ which is analytic in a neighborhood of the origin and is real for real z. Set

$$f(z) = \sum_{n=0}^{\infty} a_n z^n; \quad a_n \text{ real, } a_0 \neq 0. \tag{5.3}$$

Consider

$$\begin{aligned} |f(re^{i\theta})|^2 &= \sum_{n=0}^{\infty} r^n \sum_{k=0}^{n} a_k a_{n-k} \cos(n-2k)\theta, \quad r \text{ real}, \\ &= \sum_{n=0}^{\infty} r^n p_n(\cos\theta). \end{aligned} \tag{5.4}$$

Fejér showed that under appropriate conditions on a_k, the zeros of $p_n(x)$ are real, lie in $-1<x<1$ and the zeros of $p_n(x)$ and $p_{n+1}(x)$ separate each other. See SZEGÖ [22], Chapter 6. These are all properties of polynomials which are orthogonal on $[-1,1]$ with respect to a positive measure, so it is natural to ask what other sets of orthogonal polynomials are generated by (5.3) and (5.4). One example is

$$f(z) = (1-z)^{-\lambda} \tag{5.5}$$

and the polynomials are the ultraspherical polynomials $C_n^\lambda(x)$. The attentive reader can probably guess the answer as to what other orthogonal polynomials there are, but I will save the answer until we derive it. The first question that occurs is how can one start to answer such a question. Clearly the orthogonality is the key, and there is another property which is equivalent to orthogonality.

We say $\{p_n(x)\}$, $p_n(x)$ of exact degree n, is orthogonal if

$$\int_a^b p_n(x)p_m(x)d\alpha(x) = 0, \qquad m \neq n \qquad (5.6)$$
$$= h_n^{-1} > 0, \quad m = n,$$

where $d\alpha(x)$ is a positive measure. The measure need not be unique, and the moment problem attached to the special case $\beta = \infty$ of Ramanujan's discrete q-beta integral on $[0, \infty)$ is an instance. Ramanujan evaluated the integral obtained when $d_q t$ is replaced by dt in (4.9), and its special case $\beta = \infty$ has the same moments as (4.9) has when $\beta = \infty$. See [5] for an evaluation of Ramanujan's other q-beta integral on $(0, \infty)$.

Every set of orthogonal polynomials $\{p_n(x)\}$ satisfies a three-term recurrence relation of the form

$$xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x), \qquad (5.7)$$

with A_n, B_n, C_n real and $A_{n-1}C_n > 0$, $n = 1, 2, \ldots$. This is immediate from orthogonality. The converse is also true. If $\{p_n(x)\}$ satisfies (5.7) with real coefficients and $A_{n-1}C_n > 0$, $n = 1, 2, \ldots$, then there is a positive measure $d\alpha(x)$ for which (5.6) holds. This is an old theorem, which was rediscovered a few times before people began to notice it. It was first brought to general attention by Favard's discovery of it in 1935 [9], and that is probably what led Feldheim and Lanzewizky to this problem. Since $p_n(x)$ defined in (5.4) satisfies

$$p_n(-x) = (-1)^n p_n(x), \qquad (5.8)$$

$B_n = 0$. We then rewrite (5.7) as

$$2\cos\theta p_n(\cos\theta) = A_n p_{n+1}(\cos\theta) + C_n p_{n-1}(\cos\theta). \qquad (5.9)$$

Using (5.4) in (5.9) we get

$$\begin{aligned} \sum_{k=0}^{n} a_k a_{n-k} \cos(n+1-2k)\theta + \sum_{k=0}^{n} a_k a_{n-k} \cos(n-1-2k)\theta \\ = A_n \sum_{k=0}^{n+1} a_k a_{n+1-k} \cos(n+1-2k)\theta \\ + C_n \sum_{k=0}^{n-1} a_k a_{n-1-k} \cos(n-1-2k)\theta. \end{aligned} \qquad (5.10)$$

Then the coefficients of $\cos(n+1)\theta$ give

$$2a_0 a_n = 2A_n a_0 a_{n+1} \qquad (5.11)$$

since $\cos(n+1)\theta = \cos(-n-1)\theta$. Thus

$$A_n = a_n/a_{n+1} \quad \text{since } a_0 \neq 0\,. \tag{5.12}$$

The coefficients of the next two terms give

$$a_1 a_{n-1} + a_0 a_n = \frac{a_n}{a_{n+1}} a_1 a_n + C_n a_0 a_{n-1} \tag{5.13}$$

and

$$a_2 a_{n-2} + a_1 a_{n-1} = \frac{a_n}{a_{n+1}} a_2 a_{n-1} + C_n a_1 a_{n-2}\,. \tag{5.14}$$

These give C_n, and when these values of C_n are equated and simplified, using

$$S_n = \frac{a_n}{a_{n-1}}\,, \tag{5.15}$$

the result is

$$S_{n+1}[S_1 + S_n - S_2 - S_{n-1}] = S_1 S_n - S_2 S_{n-1}\,. \tag{5.16}$$

Set

$$S_n = T_n + S_1, \quad \text{so } T_1 = 0\,. \tag{5.17}$$

The equation (5.16) becomes

$$\begin{aligned}(T_{n+1} + S_1)(T_n - T_{n-1} + S_1 - S_2)\\ = S_1(T_n + S_1) - S_2(T_{n-1} + S_1)\end{aligned} \tag{5.18}$$

or

$$T_{n+1}(T_n - T_{n-1}) = (S_2 - S_1)(T_{n+1} - T_{n-1})\,. \tag{5.19}$$

This gives T_{n+1} as a rational function of T_n and T_{n-1}, so the most elementary function we could hope for as a solution of (5.19) is a rational function of something. Since $T_1 = 0$ it is natural to try

$$T_n = \frac{A(1 - q^{n-1})}{(1 - Bq^n)}\,. \tag{5.20}$$

Using this in (5.19) gives

$$A(1 - q^n) = (S_2 - S_1)(1 + q)(1 - Bq^n)\,. \tag{5.21}$$

Then $A = (S_2 - S_1)(1 + q)$, $B = 1$ when $|q| \neq 1$. This gives

$$S_n = T_n + S_1 = \frac{\alpha(1 - \beta q^{n-1})}{1 - q^n} \tag{5.22}$$

for constants α and β. Then

$$a_n = \frac{\alpha^n (\beta; q)_n}{(q; q)_n} \tag{5.23}$$

and when $|q| < 1$,

$$f(z) = \sum_{n=0}^{\infty} \alpha^n \frac{(\beta : q)_n}{(q; q)_n} z^n = \frac{(\beta \alpha z; q)_\infty}{(\alpha z; q)_\infty}, \tag{5.24}$$

by the q-binomial theorem. The constant α is a scale factor which can be taken to be 1. The orthogonal polynomials are now denoted by $C_n(x; \beta|q)$ and their three-term recurrence relation is

$$\begin{aligned} 2x(1 - \beta q^n) C_n(x; \beta|q) &= (1 - q^{n+1}) C_{n+1}(x; \beta|q) \\ &\quad + (1 - \beta^2 q^{n-1}) C_{n-1}(x; \beta|q) . \end{aligned} \tag{5.25}$$

When $|q| < 1$ all of the above is correct as given. If $|q| > 1$ it is possible to carry out the same type of argument since

$$S_n = \frac{\alpha\beta}{q} \frac{(1 - (q^{-1})^{n-1} \beta^{-1})}{(1 - q^{-n})} \tag{5.26}$$

reduces this case to the case when $|q| < 1$. In addition to the case when $\beta = q^\lambda$ as $q \to 1$, which gives the ultraspherical polynomials $C_n^\lambda(x)$ with $f(z) = (1 - z)^{-\lambda}$, there is a case when q is a root of unity. The orthogonality for all these cases have finally been worked out. For $-1 < q < 1$ there are a number of treatments and an extension to a more general set of orthogonal polynomials with four free parameters in addition to the q. See GASPER and RAHMAN [13] for references and treatments of some of these. For q a root of unity, ALLAWAY [3] was the first to realize that some interesting orthogonal polynomials arise in this case. He was treating a different characterization problem and rediscovered the continuous q-ultraspherical polynomials of Rogers along with a different limiting case when q is a root of unity. Both of these two cases when q is a root of unity are considered in [2], and the orthogonality is worked out there.

To complete this short introduction to q-series, the integral which can be used to prove the orthogonality of $C_n(x; \beta|q)$ will be evaluated. It is

$$I_k = \int_0^\pi e^{-2ik\theta} \frac{(e^{2i\theta}; q)_\infty (e^{-2i\theta}; q)_\infty}{(\beta e^{2i\theta}; q)_\infty (\beta e^{-2i\theta}; q)_\infty} \, d\theta$$

when $|\beta| < 1$. To evaluate I_k observe that the infinite products almost can be matched with the factors depending on x in Ramanujan's sum (3.21). The only change needed is to take off the factor $(1 - e^{-2i\theta})$ so that the numerator factors are a theta product. Then with $x = \beta e^{2i\theta}$, $a = \beta^{-1}$, $b = \beta$, we have

$$
\begin{aligned}
I_k &= \frac{(\beta;q)_\infty(\beta q;q)_\infty}{(q;q)_\infty(\beta^2;q)_\infty}\sum_{-\infty}^{\infty}\frac{(\beta^{-1};q)_n}{(\beta;q)_n}\beta^n\int_0^\pi e^{-2ik\theta}(1-e^{-2i\theta})e^{2in\theta}d\theta \\
&= \pi\frac{(\beta;q)_\infty(\beta q;q)_\infty}{(q;q)_\infty(\beta^2;q)_\infty}\;\frac{(\beta^{-1};q)_k\beta^k(1+q^k)}{(\beta q;q)_k}.
\end{aligned} \tag{5.27}
$$

The rest of the proof of the orthogonality from this integral is given in [6]. The orthogonality for $-1 < \beta,\ q < 1$ is

$$
\begin{aligned}
\int_{-1}^{1} & C_n(x;\beta|q)C_m(x;\beta|q)\cdot \\
&\cdot\prod_{k=0}^{\infty}\left[\frac{1-2(2x^2-1)q^k+q^{2k}}{1-2(2x^2-1)\beta q^k+\beta^2q^{2k}}\right]\frac{dx}{(1-x^2)^{\frac12}} = 0,\ m\neq n \\
&= 2\pi\frac{(1-\beta)(\beta^2;q)_n(\beta;q)_\infty(\beta q;q)_\infty}{(1-\beta q^n)(q;q)_n(\beta^2;q)_\infty(q;q)_\infty},\ m=n.
\end{aligned} \tag{5.28}
$$

I have not mentioned the reason for the use of the vertical bar in the notation of $C_n(x;\beta|q)$. There are two different q-ultraspherical polynomials. The other [4] is orthogonal with respect to a discrete measure symmetrically located about $x=0$ with the masses supported on two geometric progressions with 0 as the limit point. These are denoted by $C_n(x;\beta:q)$. As architects say, form should follow function, so the dots denote the discrete case and the bar the absolutely continuous case.

All of this work comes up in the study of $SU_q(2)$ and related quantum groups. The biggest surprise, after the original discovery of the connection with quantum groups and basic hypergeometric series, was the discovery that the integral (5.27) and the orthogonality (5.28) contain special cases which arise on $SU_q(2)$. This was discovered by KOORNWINDER [17].

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